E2.5 Signals & Linear Systems

Tutorial Sheet 2 Zero-input Responses

Solutions

1. System equation is

$$(D^2 + 4D + 4)\, y(t) = D f(t).$$

Characteristic polynomial is:

$$\lambda^2 + 4\lambda + 4$$

Characteristic equation is:

$$\lambda^2 + 4\lambda + 4 = 0 = (\lambda+2)^2.$$

This has repeated roots of $\lambda = -2$ (twice).

For repeated roots, characteristic modes are e^{-2t} and te^{-2t}.

Therefore, zero-input response $y_0(t)$ is

$$y_0(t) = c_1 e^{-2t} + c_2 t e^{-2t} \; ; \; \dot{y}_0(t) = -2c_1 e^{-2t} - 2c_2 t e^{-2t} + c_2 e^{-2t}$$

Letting $t=0$, substituting initial conditions gives:–

$$\left.\begin{aligned} 3 &= c_1 \\ -4 &= -2c_1 + c_2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 3 \\ c_2 &= 2. \end{aligned}$$

$$\therefore \; \underline{\underline{y_0(t) = (3 + 2t)\, e^{-2t}}}$$

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2. System equation is $D(D+1)\,y(t) = (D+2)\,f(t)$.

Characteristic polynomial is

$$\lambda(\lambda+1) = \lambda^2 + \lambda.$$

Characteristic equation is

$$\lambda(\lambda+1) = 0.$$

Characteristic ~~mode~~ roots are 0 and -1.

Characteristic modes are 1 and e^{-t}.

Therefore

$$y_0(t) = c_1 + c_2 e^{-t}$$

$$\dot{y}_0(t) = -c_2 e^{-t}$$

At $t=0$, $y(0)=1$, $\dot{y}(0)=1$.

$$\therefore \left.\begin{aligned} 1 &= c_1 + c_2 \\ 1 &= -c_2 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} c_1 &= 2 \\ c_2 &= -1. \end{aligned}$$

$$\therefore \quad y_0(t) = 2 - e^{-t}$$

$\boxed{S2.2}$

3. System equation is

$$(D^2+9)\,y(t) = (3D+2)\,f(t).$$

Characteristic equations & roots are :-

$$\lambda^2 + 9 = 0, \quad \Rightarrow (\lambda + j3)(\lambda - j3) = 0.$$

$\therefore \lambda = \pm j3$

Characteristic modes are :-

$$e^{j3t}, \quad e^{-j3t}. \quad \text{(Complex roots)}$$

$$\therefore \quad y_0(t) = c\cos(3t + \theta)$$

and $\dot{y}_0(t) = -3c\sin(3t+\theta)$.

Given at $t=0$, $y_0(0) = 0$, $\dot{y}_0(0) = 6$,

$$\left.\begin{aligned} 0 &= c\cos\theta \\ 6 &= -3c\sin\theta \end{aligned}\right\} \Rightarrow \left.\begin{aligned} c\cos\theta &= 0 \\ c\sin\theta &= -2 \end{aligned}\right\} \Rightarrow \begin{aligned} c &= 2 \\ \theta &= -\frac{\pi}{2}. \end{aligned}$$

Therefore

$$y_0(t) = 2\cos\left(3t - \frac{\pi}{2}\right)$$

$$= \underline{\underline{2\sin 3t}}$$

$\boxed{S\ 2.3}$

4. a) Remember $\delta(x)$ is located at $x=0$,
and $\delta(t-\tau)$ is located at $\tau=t$, etc.

Therefore ~~δτ~~ impulse is at $\tau=t$,
and $f(\tau)$ at $\tau=t$ is $f(t)$.
Therefore
$$\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau = f(t).$$

b) ~~Since~~ Similarly $\delta(\tau)$ is at $\tau=0$,
$f(t-\tau)$ at $\tau=0$ is $f(t)$.
$$\therefore \int_{-\infty}^{\infty} \delta(\tau)\, f(t-\tau)\, d\tau = f(t).$$

c) $$\int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t} dt = e^{j\omega \emptyset} = 1.$$

d) $$\int_{-\infty}^{\infty} \delta(t-2) \sin \pi t \, dt$$

$$= \sin 2\pi = \emptyset.$$

$\boxed{S \cdot 2 \cdot 4}$

5. The characteristic equation is

$$\lambda^2 + 4\lambda + 3 = (\lambda+1)(\lambda+3) = 0.$$

Characteristic modes are e^{-t} and e^{-3t}.

Therefore

$$y_o(t) = c_1 e^{-t} + c_2 e^{-3t}$$

$$\dot{y}_o(t) = -c_1 e^{-t} - 3c_2 e^{-3t}.$$

At $t=0$, $y(0)=0$, $\dot{y}(0)=1$ (See notes);

we get:

$$\left.\begin{aligned} 0 &= c_1 + c_2 \\ 1 &= -c_1 - 3c_2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= \tfrac{1}{2} \\ c_2 &= -\tfrac{1}{2} \end{aligned}$$

Therefore

$$y_o(t) = \frac{1}{2}\left(e^{-t} - e^{-3t}\right)$$

$$\begin{aligned} h(t) &= [P(D)\, y_o(t)]\, u(t) \\ &= [(D+5)\, y_o(t)]\, u(t). \\ &= [\dot{y}_o(t) + 5\, y_o(t)]\, u(t) \\ &= (2e^{-t} - e^{-3t})\, u(t) \end{aligned}$$

Also check that order of $Q(D) = 2$ (i.e. N) and order of $P(D) = \underline{1}$, $\therefore$ $N > M$.

$\boxed{\text{S 2.5}}$

6. The system equation is more complex here:

$$(D+1)(D^2+5D+6)\,y(t) = (5D+9)\,f(t).$$

Characteristic equation:

$$(\lambda+1)(\lambda^2+5\lambda+6) = 0$$

$$\Rightarrow (\lambda+1)(\lambda+2)(\lambda+3) = 0.$$

Therefore

$$y_0(t) = c_1 e^{-t} + c_2 e^{-2t} + c_3 e^{-3t}.$$

$$\dot{y}_0(t) = -c_1 e^{-t} - 2c_2 e^{-2t} - 3c_3 e^{-3t}$$

$$\ddot{y}_0(t) = c_1 e^{-t} + 4c_2 e^{-2t} + 9c_3 e^{-3t}$$

Since $N = 3 > M = 1$, at $t=0$, $y_0(0) = \dot{y}_0(0) = 0$,
$\ddot{y}(0) = 1$.

$\therefore$ We obtain:

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 0 &= -c_1 - 2c_2 - 3c_3 \\ 1 &= c_1 + 4c_2 + 9c_3 \end{aligned} \quad \Rightarrow \quad \begin{aligned} c_1 &= \tfrac{1}{2} \\ c_2 &= -1 \\ c_3 &= \tfrac{1}{2} \end{aligned}$$

$$\therefore y_0(t) = \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}$$

Finally,

$$\begin{aligned} h(t) &= [P(D)\,y_0(t)]\,u(t) \\ &= [5\dot{y}_0(t) + 9y_0(t)]\,u(t) \\ &= (2e^{-t} + e^{-2t} - 3e^{-3t})\,u(t) \end{aligned}$$

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